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Примена Питагорине теореме на темпоралност ритмичких пројекција појединачних хармоника спектра

The application of the Pythagorean Theorem to the temporality of rhythmical projections of individual spectrum harmonics

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Abstract

The procedure of the summation of metro-rhythmical structures, derived by means of spectrum projection, which could potentially be proved through the Pythagorean Theorem, has been described in this study. The aim of this work is to identify the length of sides and trigonometric values of the angles of the right-angled spectral triangles with the help of known parameters of original and projected spectral entities. The connection between the intervals of the interval fund of the sixth aliquot of the spectrum which includes an *octave* (1/2), *a twelfth* (2/3) and *a super twelfth* (5/6), which has been established through the Pythagorean Theorem, and which can be described with the following mathematical relation: $(1/2)^2 + (2/3)^2 = (5/6)^2$, will be applied to the interval funds of the other individual spectrum harmonics in the continuation of the work.

Key words:
Kahun papyrus, rhythm, intervals, trigonometry, sound spectrum

Сажетак

У овом раду описан је поступак сумације метро-ритмичких структура, изведених посредством пројекције спектра, који би потенцијално био доказив путем Питагорине теореме. Циљ рада је да се уз помоћ познатих параметара изворних и пројектованих спектралних ентитета, идентификују дужине страница и тригонометријске вредности углова правоуглих спектралних троуглова. Веза интервала интервалског фонда шестог аликвота спектра којег чине: *октава* (1/2), *дуодецима* (2/3) и *супердуодецима* (5/6), успостављена посредством Питагорине теореме, а што се може описати следећом математичком релацијом: $(1/2)^2 + (2/3)^2 = (5/6)^2$, у наставку рада биће примењена и на интервалске фондове осталих појединачних хармоника спектра.

Кључне речи:
папирус из Кахуна, ритам, интервали, тригонометрија, звучни спектар

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Introduction

The old Egyptians knew about four Pythagorean triplets, which is testified by the papyrus, dating back to the times of the 12th dynasty reign, around 2000 years before Christ, in which the following relation, among other things, is mentioned: $1^2 + (3/4)^2 = (1\ ¼)^2$: "The relation of the sides of a triangle when these sides are 3, 4 and 5 (that is $3^2 + 4^2 = 5^2$) was well known before the time of Pythagoras. We find in the *Nine Sections* of the Chinese, perhaps written before 1100 B.C., this statement: 'Square the first side and the second side and add them together; then the square root is the hypotenuse.' The Egyptians also knew the numerical relation for special cases, for a papyrus of the 12th dynasty (c. 2000 B.C.), discovered at Kahun, refers to four of these relations, one being $1^2 + (3/4)^2 = (1¼)^2$. It was among those people that we first heard of the 'rope stretchers', those surveyors who, as usually thought, were able by the aid of this property to stretch a rope so as to draw a line perpendicular to another line, a method still in use at the present time." [4]

The above-mentioned mathematical relation: $1^2 + (3/4)^2 = (1¼)^2$, found in the papyrus of the old Egyptians, will not be connected to the 'rope stretching' in the further text, but it will be, in a certain way, connected with the spectrum law. Having in mind that rhythmic chords, to which we came through the projection of the original relations of individual spectrum harmonics, were identified with some elementary geometric triangle figures, and polygons, the time has come to prove this type of identification. One of the numerous consequences of this experiment will appear, among other things, in the mathematical equation – the equation that is identical to that one from the above-mentioned papyrus.

The Theorem of the referent minor third right-angled triangle

"Relations, which were valid for interval constellations of the interval fund of the third aliquot, and these are 1/3 + 2/3 and 2/3 + 1/3 and with their superposition could establish a unique relation: 1/3 + 1/3 + 1/3. […] We state that the projected accord makes the triangle on the flat surface which is in the reference system of the third aliquot – equilateral triangle (all angles are the same) – $60^0 + 60^0 + 60^0$." [2, 3] In the reference system of the sixth harmonic, the right-angled triangles, whose relation of the angles is $90^0\ 60^0\ 30^0$, are seen for the first time. Constellation $6^0{:}5^0{:}3^0$ will be taken for the research we deal with – a chord, which is made of a minor third ($6^0{:}5^0$), a major sixth ($5^0{:}3^0$) and an octave ($6^0{:}3^0$). The value of the foundation for this experiment, and for the following experiments, in the projection is a rhythmical fourth. The rhythmical construction of the projected chord is: (1) one triolet sixteenth (a minor third), (2) one triolet eighth (a perfect fifth) and (3) one rhythmical eighth (octave). At first sight, it seems that the rhythmical eighth, which is the longest in the projected constellation, was conceived as the "hypotenuse", and that the triolet sixteenth and triolet eighth, which are two shorter lengths, were conceived as legs of a triangle (the first would be shorter, whereas the second would be longer). The Pythagorean Theorem is proved by the binary operation – rising to the power, the operation for which, besides the well-known exponent ("the square of"), we also need the base, that is, the common denominator of the projected construction for all three lengths (sides). The triolet sixteenth would be a common denominator for the triolet eighth as well (2 triolet sixteenths) and for the rhythmical eighth (3 triolet sixteenths), as well as for itself (one triolet sixteenth). Through the first Pythagorean triplet which includes the numbers: |3 4 5|, the triolet sixteenth, as a common denominator, would be multiplied through the summation of rhythmical constituents |1 2 3|, in the following way: (1+2) (1+3) and (2+3).

Thus, the adjacent leg of a triangle would contain three triolet sixteenths (1+2), and therefore, its whole length would amount to one rhythmical eighth. *Umbra* of partial octave – ($2^0{:}1^0$) is marked by the rhythmical eighth, in the value of one-half of the foundation (1/2).

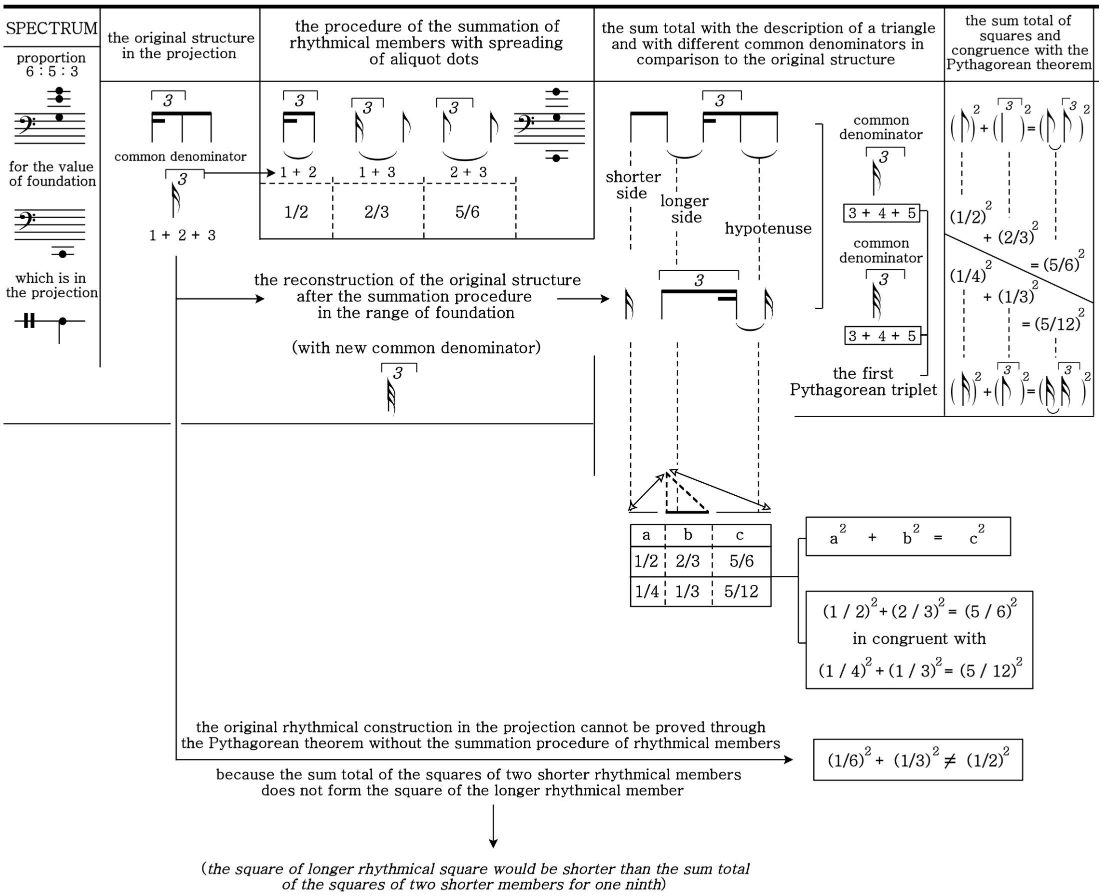
The opposite side of a triangle would contain four triolet sixteenths (1+3), and therefore, the whole length of this side would amount to one triolet fourth. *Umbra* of partial twelfth is marked with triolet fourth – (3^0:1^0), in the value of two-thirds of the foundation (2/3). Hypotenuse, as the longest side of the right-angled triangle, would contain five triolet sixteenths (2+3), and therefore, its total length would amount to a rhythmical eighth (1/2) bound with the triolet eighth (1/3) which in total give the fraction: (5/6). Umbra of partial super twelfth – (6^0:1^0) is marked with the rhythmical eighth bound with the triolet eighth, in value five/sixths of the foundation (5/6).

It is important to mention that through the summation of rhythmical constituent members, the process of spreading aliquot dots in spectrum took place in parallel. Newly formed harmonic construction (with spreading of aliquot dots 6^0:5^0:3^0) would contain the following measuring relations: (6^0:3^0:1^0) – thus, this construction would contain *an octave* (6^0:3^0 or 2^0:1^0), *a twelfth* (3^0:1^0) and *a super twelfth* (6^0:1^0). Regarding the fact that we have come to the newly gained rhythmical construction through summation – the construction whose value would amount to the twofold foundation line (foundation = 2), the check of the measuring relations for the real value of the foundation line (foundation = 1) would be performed by the diminution in the metrical octave (twofold reduction of rhythmical values). In that way, the following construction would be formed: a rhythmical sixteenth (1/4 of the foundation), a triolet eighth (1/3 of the foundation) and a triolet sixteenth bound with the rhythmical sixteenth (5/12 of the foundation). We notice that the newly formed constellation is not the same compared to the original (6^0:5^0:3^0). The common denominator is chosen from the biggest denominator of the offered fractions – which is one-twelfth of the foundation (1/12) whose value is one triolet thirty-two. It is the twelfth harmonic, and the relation of harmonics in the reference system of the twelfth harmonic would be the following: (12^0:9^0:5^0).

If we get back to the original construction of the rhythmical constellation in question (6^0:5^0:3^0), we will notice that the projected rhythmical intervals are shown as the sides of a right-angled triangle through summation, as follows.

1. The sum of a triolet sixteenth (1/6) and a triolet eighth (1/3) formed the rhythmical eighth, that is, the length of a metrical octave (1/2 of the foundation) which, as the shortest length, can be named <u>the adjacent “octave” side</u> (a).
2. The sum of the triolet sixteenth (1/6) and rhythmical eighth (1/2) formed the triolet fourth, that is, the length of a metrical twelfth (2/3 of the foundation) which, as the middle length (longer than the previous, but shorter than the following), can be named <u>the opposite “twelfth” side</u> (b).
3. The sum of the triolet eighth (1/3) and rhythmical eighth (1/2) formed the rhythmical eighth bound with the triolet eighth, that is, the length of a metrical super twelfth (5/6 of the foundation) which, as the longest length, can be named <u>“super twelfth” hypotenuse</u> (c).

With this procedure, it can be stated that the credibility of the musical notation is in accordance with the picture of the right-angled triangle in the simple (original) projection, but it is more relevant that the sum total of rhythmical constituent members of this original projection coincides with the mathematical relation of the Pythagorean Theorem. It is more relevant due to the fact that only through the summation of rhythmical members binary operation can be accomplished, which would be necessary for the application of the Pythagorean Theorem to the given rhythmical constellation. The sum total of common denominators from the original structure (1+2+3) would not coincide with the operation of the Pythagorean Theorem because the numbers with which we present common denominators are not the numbers of the Pythagorean triplet. However, with the above-described summation of rhythmical members (1+2) + (1+3) + (2+3), the constellation, which could potentially be proved through the Pythagorean Theorem, would be formed, thus keeping the value of the common denominator.



Thus, "the square of octave", whose common denominator is a triolet sixteenth, and which through the number 3 (the first number of the Pythagorean triplet) formed the octave adjacent side, would amount to nine triolet sixteenths. We can state that *the square of an octave* is in fact – *summation rhythmical twelfth* (3/2).

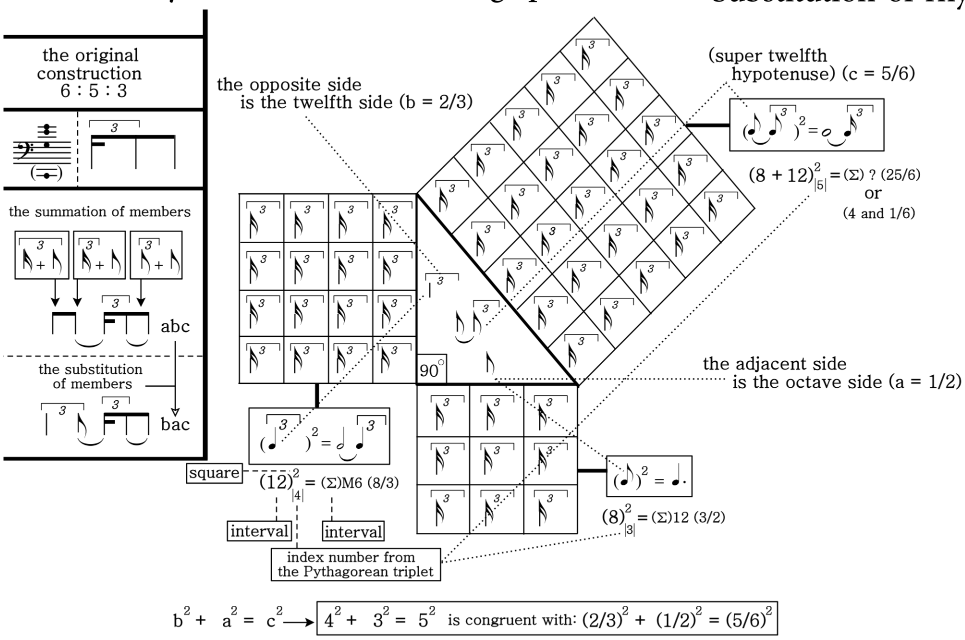
"The square of a twelfth", whose common denominator is a triolet sixteenth, and which through the number 4 (the second number of the Pythagorean triplet) formed the twelfth opposite side, would amount to sixteen triolet sixteenths. We can state that *the square of a twelfth* is in fact – *summation rhythmical minor sixth* (8/3).

Finally, "the square of super twelfth" whose common denominator is a triolet sixteenth, and which through the number 5 (the third number of the Pythagorean triplet) formed super twelfth hypotenuse, would amount to twenty-five triolet sixteenths. We can state that *the square of super twelfth* is, in fact – *rhythmical whole note bound with a triolet sixteenth* (25/6), which is not, taking into consideration a high numerator, discernible in the aliquot system of natural aliquot intervals. This fraction can be expressed as 4 and 1/6 where the value of 1/6 could be identified with the differential interval of a minor third ($6^0:5^0$), thus with the interval which is a referent for the system of the sixth harmonic. Consequently, the sum total of *the rhythmical eighth* squared and *triolet fourth* squared equals the square of *one rhythmical eighth bound with the triolet eighth*. Mathematical equation for the first Pythagorean triplet: $3^2 + 4^2 = 5^2$ according to the Pythagorean Theorem $a^2 + b^2 = c^2$ in the referent system of the sixth harmonic (in which the right-angled triangle is the referent triangle) is congruent with the equation: $(1/2)^2 + (2/3)^2 = (5/6)^2$.

In the language of music, the expression would be: The sum total of the *partial octave (rhythmical eighth)* and the square of a *partial twelfth (a triolet fourth)* equals the square of one *partial super twelfth (rhythmical eighth bound with one triolet eighth*).

$(1/2)^2 + (2/3)^2 = (5/6)^2$ is congruent with:
$3/2 + 8/3 = 25/6$.

In picture 2, the procedure of the application of the Pythagorean Theorem to the right-angled triangle of the sixth aliquot with the substitution of rhythmical members has been described (b a c). Under the expression – *the substitution of rhythmical members*, we mean that the offered rhythmical values among the one and only (metrical) line – change places.



The temporality of rhythm-metrical triangles is feasible, with the extension, of course, and in the summation of higher Pythagorean triplets: |5 12 13| |8 15 17| and so on. Nevertheless, in order to prove the applicability of the Pythagorean Theorem to the referent triangle of the sixth aliquot we will use the first derived triplets from the base |3 4 5|, and they are: |6 8 10| as well as |9 12 15|, under the condition that for all of them the valid rule is that the common denominator is a triolet sixteenth and in the following way:

|1 2 3| (1+2) (1+3) (2+3) |3 4 5| is congruent with $(1/2)^2 + (2/3)^2 = (5/6)^2$

|2 4 6| (2+4) (2+6) (4+6) |6 8 10| is congruent with $1^2 + (4/3)^2 = (5/3)^2$

|3 6 9| (3+6) (3+9) (6+9) |9 12 15| is congruent with $(3/2)^2 + 2^2 = (5/2)^2$

If we read carefully the following text about the application of the Pythagorean Theorem to the octave and the fifth right-angled triangle, we will notice that mathematical relations are in accordance with the multiplication of the common denominator of the first Pythagorean triplet. Thus, we can state that the rhythmical relation of the spectrum harmonics $6^0:5^0:3^0$ can be proved through the Pythagorean Theorem due to the fact that the constellation contains a minor third and a fifth and an octave!

Substitution of rhythmical members is performed with the mathematical analysis of the trigonometric functions. Six right-angled triangles of the sixth harmonic are:

1. |a b c| or the proportion 6^0: 5^0: 3^0 harmonic (1+2) (1+3) (2+3) or |3 4 5|
2. |a c b| or the proportion 6^0: 5^0: 2^0 harmonic (1+2) (3+2) (3+1) or |3 5 4|
3. |b a c| or the proportion 6^0: 4^0: 3^0 harmonic (1+3) (2+1) (2 +3) or |4 3 5|
4. |b c a| or the proportion 6^0: 4^0: 1^0 harmonic (3+1) (2+3) (2+1) or |4 5 3|
5. |c a b| or the proportion 6^0: 3^0: 2^0 harmonic (3+2) (1+2) (3+1) or |5 3 4|
6. |c b a| or the proportion 6^0: 3^0: 1^0 harmonic (3+2) (3+1) (2+1) or |5 4 3|.

Besides the classification of the right-angled triangles of the sixth harmonic, the procedure of the summation of members, which contain one, two or three common denominators (a triolet sixteenth), is given. Somewhere we notice the equation 1+2, 1+3 and 2+3, whereas somewhere the inverse variant of this equation 2+1, 3+1 and 3+2, because the summation of members is performed by the order how the members of that constellation are juxtaposed, with the aim not to disrupt the order of the original structure in the summated structure. All of these right-angled triangles contain six basic trigonometric functions: sine, cosine, tangent, cotangent, secant and cosecant.

We define dots on the straight line, which will divide three its segments in the relation: $2^0:1^0$, $3^0:1^0$ and $6^0:1^0$. We notice the segment of *the octave line* (8), the segment of *the twelfth line* (12) and the segment of *the super twelfth* (8+12). If we straighten the segment line of the octave for 90^0, the starting point moves to the height, and the segment line of the octave becomes the adjacent side. We do not touch the segment line of the twelfth. The last dot of the segment line of super twelfth (which is at the same time the last dot of the completely straight line) is bound with the first – by which the top of the newly gained triangle is formed. Since we know that the longest segment is the super twelfth side, we will name it hypotenuse, whereas the still segment (side of a twelfth) we will name the adjacent side.

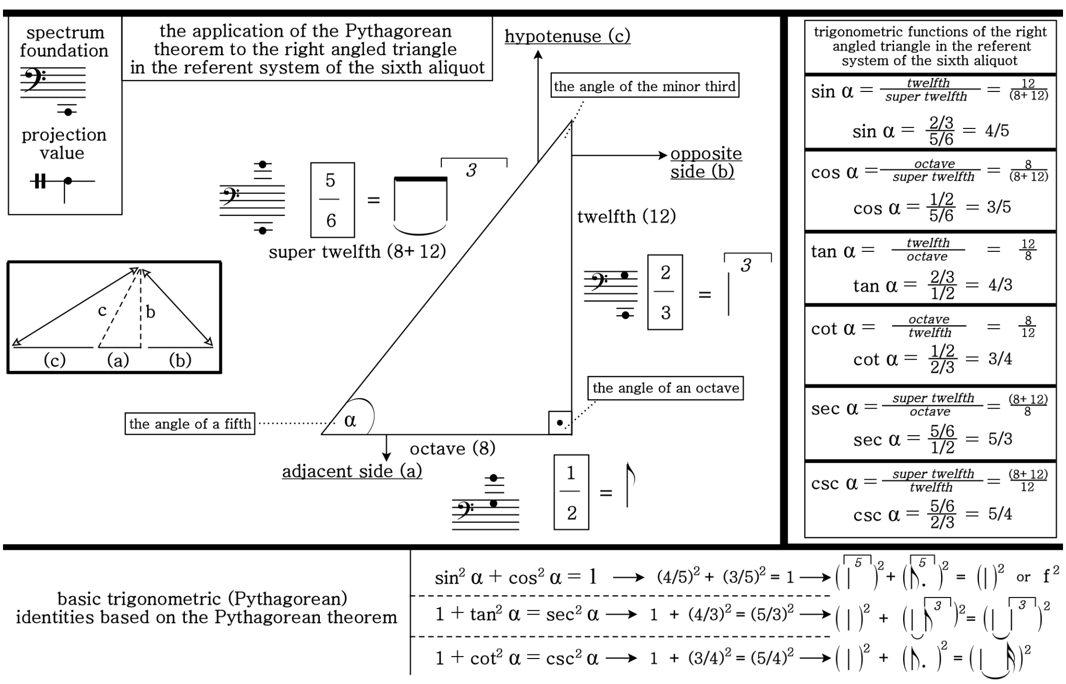
The angle of the octave and twelfth side amounts to 90^0, the angle of the twelfth side and super twelfth hypotenuse amounts to $\theta = 30^0$, whereas the angle of the super twelfth hypotenuse and octave side amounts to the rest 60^0. We determine trigonometric functions for the referent angle of the right-angled triangle of the sixth harmonic – that is $\theta = 30^0$.

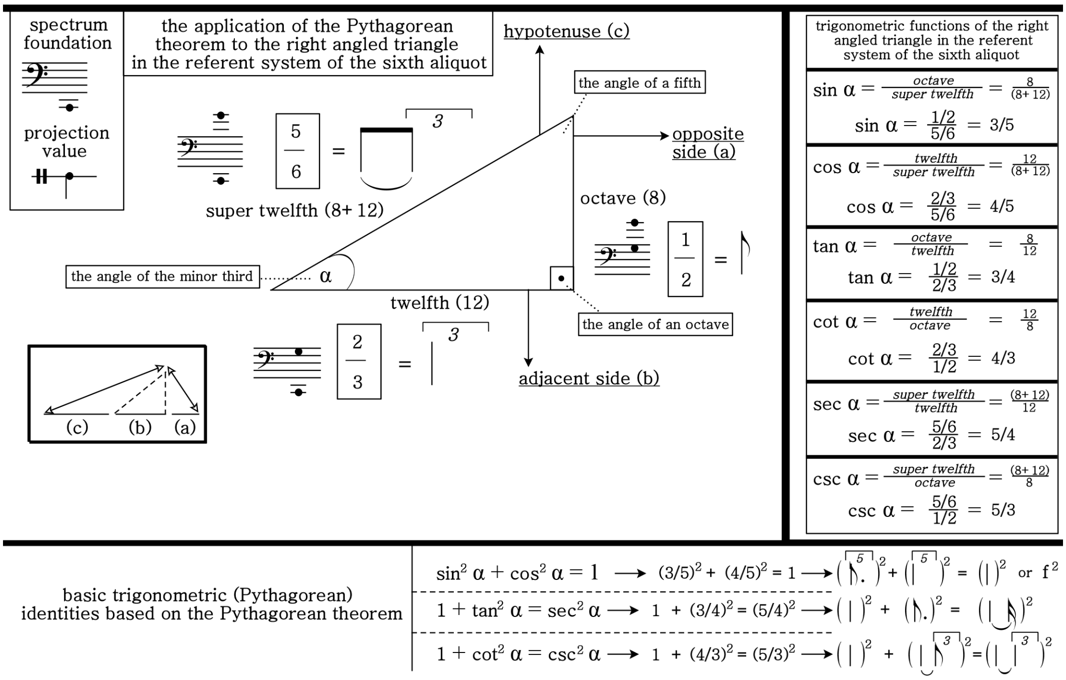
The sine of the angle – sin θ, is the relation of the opposite side and hypotenuse [1], more precisely, the quotient of an octave and super twelfth (in the language of music: 8/(8+12) which is in measuring relations (1/2)/(5/6). The quotient of these two fractions is 3/5 – thus, sin θ is the value expressed by the interval of a *major tenth*.

The cosine of the angle – cos θ, is the relation of the adjacent side and hypotenuse [1], more precisely, the quotient of twelfth and super twelfth (in the language of music 12/(8+12) which is in measuring relations (2/3)/(5/6). The quotient of these two fractions is 4/5 – thus, cos θ is the value expressed by the interval of the *major super twelfth*.

The tangent of the angle – tan θ, is the relation of the opposite and the adjacent side [1], more precisely, the quotient of an octave and a twelfth (in the language of music: 8/12 which is in measuring relations (1/2)/(2/3). The quotient of these two fractions is 3/4 – thus, tan θ is the value expressed by the interval of the *partial double octave*.

The cotangent of the angle – cot θ, is the relation of the adjacent and opposite side [1], more precisely, the quotient of a twelfth and an octave (in the language of music: 12/8 which is in measuring relations (2/3)/(1/2). The quotient of these two fractions is 4/3 – thus, cot θ is the value expressed by the interval of the *summation double octave*.





The secant of the angle – sec θ, is the relation of the hypotenuse and adjacent side [1], more precisely, the quotient of a super twelfth and a twelfth (in the language of music: (8+12)/12 which is in measuring relations (5/6)/(2/3). The quotient of these two fractions is 5/4 – thus, sec θ is the value expressed by the interval of *summation major super twelfth*.

The cosecant of the angle – csc θ is the relation of the hypotenuse and opposite side [1], more precisely, the quotient of a supper twelfth and an octave (in the language of music: (8+12)/8 which is in measuring relations (5/6)/(1/2). The quotient of these two fractions is 5/3 – thus, csc θ is the value expressed by the interval of the *summation twelfth*.

The Theorem of the octave right-angled triangle

In the referent system of the second harmonic, *umbra* of the partial octave is *one rhythmical eighth* (1/2 of the foundation) if one rhythmical fourth as a measuring whole is presented for the spectrum foundation (1). Through the first Pythagorean triplet |3 4 5|, rhythmical eighth is multiplied by the application of the Pythagorean Theorem for which the following rule is valid: $a^2 + b^2 = c^2$, and in the following way: (3 × one rhythmical eighth)^2 + (4 × one rhythmical eighth)^2 = (5 × one rhythmical eighth)^2.

Consequently, the sum total of the square of the *rhythmical fourth with the dot* and the square of the *rhythmical half* equals the square of *one rhythmical half* bound with *one rhythmical eighth*. Mathematical equation for the first Pythagorean triplet: $3^2 + 4^2 = 5^2$ in the referent system of the second harmonic is congruent with the equation: $(3/2)^2 + 2^2 = (5/2)^2$.

In the language of music, the equation would be: The sum total of the summation twelfth (*fourths with the dot*) and the square of *summation octave (rhythmical half)* equals the square of one *summation major sixth (rhythmical half bound with one rhythmical eighth).*

The Pythagorean Theorem with trigonometric functions for the right-angled triangle of the second harmonic – The Theorem of octave right-angled triangle

sin θ = *opposite* / *hypotenuse* = *twelfth* / *sixth* = (3/2) / (5/2) = 3/5

cos θ = *adjacent* / *hypotenuse* = *octave* / *sixth* = 2 / (5/2) = 4/5

tan θ = *opposite* / *adjacent* = *twelfth* / *octave* = (3/2) / 2 = 3/4 cot θ = *adjacent* / *opposite* = *octave* / *twelfth* = 2 / (3/2) = 4/3

sec θ = *hypotenuse* / *adjacent* = *sixth* / *octave* = (5/2) / 2 = 5/4

csc θ = *hypotenuse* / *opposite* = *sixth* / *twelfth* = (5/2) / (3/2) = 5/3

We notice that the values of trigonometric functions of the acute angle of the right-angled triangle of the second harmonic (30^0) are congruent with the trigonometric values of the acute angle of the right-angled triangle of the referent sixth harmonic (30^0)!

The Theorem of the fifth right-angled triangle

In the referent system of the third harmonic *umbra* of the partial fifth is *one triolet eighth* (1/3 of the foundation) if we present one rhythmical fourth as a measuring whole for the spectrum foundation (1). Through the first Pythagorean triplet |3 4 5| triolet eighth is multiplied by the application of the Pythagorean Theorem, for which the following rule is valid: $a^2 + b^2 = c^2$, and in the following way: (3 × one triolet eighth)^2 + (4 × one triolet eighth)^2 = (5 × one triolet eighth)^2.

Consequently, the sum total of the square of the *rhythmical fourth* and the square of *the rhythmical fourth bound with the triolet eighth* equals the square of one *rhythmical fourth bound with one triolet fourth.* The mathematical equation for the first Pythagorean triplet $3^2 + 4^2 = 5^2$ in the referent system of third harmonic (which would be valid for the fifth) is congruent with the expression: $1^2 + (4/3)^2 = (5/3)^2$. In the language of music, this expression would be: the sum total of the *foundation (rhythmical fourth)* and the square *of the summation of a double octave (rhythmical fourth bound with the triolet eighth)* equals the square of one *summation major tenth (rhythmical fourth bound with one triolet fourth).*

The Pythagorean Theorem with trigonometric functions for the right-angled triangle of the third harmonic on the left side of the spectrum – The Theorem of the fifth right-angled triangle

sin θ = *opposite* / *hypotenuse* = *foundation* / *tenth* = 1 / (5/3) = 3/5

cos θ = *adjacent* / *hypotenuse* = *double octave* / *tenth* = (4/3) / (5/3) = 4/5

tan θ = *opposite* / *adjacent* = *foundation* / *double octave* = 1 / (4/3) = 3/4

cot θ = *adjacent* / *opposite* = *double octave* / *foundation* = (4/3) / 1 = 4/3

sec θ = *hypotenuse* / *adjacent* = *tenth* / *double octave* = (5/3) / (4/3) = 5/4

csc θ = *hypotenuse* / *opposite* = *tenth* / *foundation* = (5/3) / 1 = 5/3

We notice that the values of the trigonometric functions of an acute angle of the fifth right-angled triangle of the third harmonic (30^0) on the left side of the spectrum are congruent with the trigonometric values of the acute angle of the right-angled triangle of the referent sixth harmonic (30^0) as well as the octave right-angled triangle!

The Theorem of the twelfth right-angled triangle

In addition, in the referent system of the third harmonic *umbra* of partial twelfth is *one triolet fourth* (2/3 of the foundation) if one rhythmical fourth is presented as a measuring whole for the spectrum foundation (1). Through the first Pythagorean triplet |3 4 5| triolet fourth is multiplied by the application of the Pythagorean Theorem, for which the following rule is valid: $a^2 + b^2 = c^2$, and in the following way: (3 × one triolet fourth)^2 + (4 × one triolet fourth)^2 = (5 × one triolet fourth)^2.

Consequently, the sum total of the square of *rhythmical half* and the square of *the rhythmical half bound with the triolet fourth* equals the square of one *rhythmical half with a dot bound with one triolet eighth.* The mathematical equation for the first Pythagorean triplet $3^2 + 4^2 = 5^2$ in the referent system of the third harmonic (which would be valid for twelfth) is congruent with the expression: $2^2 + (8/3)^2 = (10/3)^2$. In the language of music, this expression would be: the sum total of the *summation octave (rhythmical half)* and the square *of the summation of minor sixth (rhythmical half bound with the triolet fourth)* equals the square of one *summation augmented fourth (rhythmical half with a dot bound with one triolet eighth).*

The Pythagorean Theorem with trigonometric functions for the right-angled triangle of the third harmonic on the right side of the spectrum – The Theorem of the twelfth right-angled triangle

sin θ = *opposite / hypotenuse = octave / + fourth* = 2 / (10/3) = 3/5

cos θ = *adjacent / hypotenuse = sixth / + fourth* = (8/3) / (10/3) = 4/5

tan θ = *opposite / adjacent = octave / sixth* = 2 / (8/3) = 3/4

cot θ = *adjacent / opposite = sixth / octave* = (8/3) / 2 = 4/3

sec θ = *hypotenuse / adjacent = + fourth / sixth* = (10/3) / (8/3) = 5/4

csc θ = *hypotenuse / opposite = + fourth / octave* = (10/3) / 2 = 5/3

We notice that the values of the trigonometric functions of an acute angle of the twelfth right-angled triangle of the third harmonic (30^0) on the right side of the spectrum are congruent with the trigonometric values of the acute angle of the right-angled triangle of the referent sixth harmonic (30^0) as well as octave right-angled triangle and fifth right-angled triangle on the left side of the spectrum!

The Theorem of the fourth right-angled triangle

In the referent system of the fourth harmonic *umbra* of the partial fourth is *one rhythmical sixteenth* (1/4 of the foundation) if one rhythmical fourth is presented as a measuring whole for the spectrum foundation (1). Through the first Pythagorean triplet |3 4 5| rhythmical sixteenth is multiplied by the application of the Pythagorean Theorem, for which the following rule is valid: $a^2 + b^2 = c^2$, and in the following way: (3 × one rhythmical sixteenth)^2 + (4 × one rhythmical sixteenth)^2 = (5 × one rhythmical sixteenth)^2.

Consequently, the sum total of the square of *rhythmical eighth with the dot* and the square of *the rhythmical fourth* equals the square of one *rhythmical fourth bound with one rhythmical sixteenth.* The mathematical equation for the first Pythagorean triplet $3^2 + 4^2 = 5^2$ in the referent system of the fourth harmonic (which would be valid for fourth) is congruent with the expression: $(3/4)^2 + 1^2 = (5/4)^2$.

This expression is identical to the expression from the old Egyptian papyrus, made 2000 years before Christ.

In the language of music this expression would be: the sum total of the *partial double octave, that is, super octave (eighth with a dot)* and the square *of the foundation (rhythmical fourth)* equals the square of one *summation major super tenth (rhythmical fourth bound with one rhythmical sixteenth).*

The Pythagorean Theorem with trigonometric functions for the right-angled triangle of the fourth harmonic on the left side of the spectrum – the Theorem of the fourth right-angled triangle

sin θ = *opposite / hypotenuse = double octave / super tenth* = (3/4) / (5/4) = 3/5

cos θ = *adjacent / hypotenuse = foundation / super tenth* = 1 / (5/4) = 4/5

tan θ = *opposite / adjacent = double octave / foundation* = (3/4) / 1 = 3/4

cot θ = *adjacent / opposite = foundation / double octave* = 1 / (3/4) = 4/3

sec θ = *hypotenuse / adjacent = super tenth / foundation* = (5/4) / 1 = 5/4

csc θ = *hypotenuse / opposite = super tenth / double octave* = (5/4) / (3/4) = 5/3

We notice that the values of the trigonometric functions of an acute angle of the fourth right-angled triangle of the fourth harmonic (30^0) on the left side of the spectrum are congruent with the trigonometric values of the acute angle of the right-angled triangle of the referent sixth harmonic (30^0) as well as: octave right-angled triangle of the second harmonic and fifth and twelfth right-angled triangle of the third harmonic on both sides of the spectrum!

Instead of a conclusion

By analogy with the lower, the applicability of the Pythagorean Theorem is possible in higher referent systems in comparison to that for which we say that it is valid as the original mathematical equation, and that is the referent system of the sixth harmonic. When we say original, it is thought that it is applicable, primarily, to the right-angled triangle with the following values of angles: 90^0 60^0 30^0, and these would, exactly, be proportions, comprehensible for the system of the sixth harmonic of the spectrum, and not for all in this system, but only for the mentioned: $6^0{:}5^0{:}3^0$, $6^0{:}5^0{:}2^0$, $6^0{:}4^0{:}3^0$, $6^0{:}4^0{:}1^0$, $6^0{:}3^0{:}2^0$ and $6^0{:}3^0{:}1^0$. In the (higher) referent system of the seventh harmonic (and others), the applicability of the Pythagorean Theorem through the first Pythagorean triplet would be valid, first of all, for the partial remainder (1/7, 2/7 and 3/7; 1/8 and 3/8 etc.), which do not exceed the value of one half, due to the fact that for the values which exceed the half of the foundation (4/7, 5/7 and 6/7; 5/8 and 7/8) the square of hypotenuse would form "high" fraction constellations which could not be identified numerically with natural rhythmical intervals (but solely with differential).

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